JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 67 (1996) 207–217

# On an system of "classical" polynomials of simultaneous orthogonality

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Received 25 October 1993; revised 1 July 1994

**Abstract**

We introduce a system of "classical" polynomials of simultaneous orthogonality, study the differential equation of third order, recurrence relation and precise the ratio asymptotic and zeros distribution of polynomials.

*Keywords:* Hermite–Padé polynomials; Simultaneous orthogonality; Difference equations of third order; Differential equations of third order

## 1. Introduction

Let $\mu_1, \mu_2, \ldots, \mu_p$ be positive Borel measures on the real line with the supports $\Delta_1, \Delta_2, \ldots, \Delta_p$ and $\bar{n} = (n_1, n_2, \ldots, n_p)$ a vector-index ($n_j \geqslant 0$). The polynomial of simultaneous orthogonality $Q_{\bar{n}}$ associated with $\bar{n}$ is defined by the following simultaneous orthogonality relations:

$$\int_{\Delta_j} Q_{\bar{n}}(x) x^k \, \mathrm{d}\mu_j(x) = 0, \quad k = 0, 1, n_j - 1, j = 1, 2, \ldots, p, \tag{1}$$

where $\deg Q_{\bar{n}} \leqslant n := n_1 + n_2 + \cdots + n_p$. The relations (1) are well known in simultaneous rational (Hermite–Padé) approximations for a systems of Markov functions (see for example [21, 10, 18, 19]). The polynomials $Q_{\bar{n}}$ are also known as vector orthogonal polynomials [12, 24]. For $p = 1$ we have the usual orthogonal polynomials. In this case one class of polynomials is of special interest. This is the class of classical orthogonal polynomials well known by their applications. Recently new classes of "classical" orthogonal polynomials were introduced and investigated in connection with

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[1] An important part of this work was made during the stay of first author at I.N.S.A. de Rouen, France.


*SSDI* 0377-0427(94)00129-4

some physical models (see the surveys [20, 2, 8] and the references there). The term "classical" is usually associated with some specific properties of orthogonal polynomials: differential equation of second order, Pearson equation for the weight-function, recurrence relation for the derivatives of polynomials and so forth. In the case of polynomials of simultaneous orthogonality there is nogeneral agreement of what is "classical". For $p = 2$ one class of "classical" polynomials of simultaneous orthogonality was introduced in [11]. The polynomials of [11] are orthogonal with respect to linear functionals defined on the set of polynomials, but these functionals are not positive definite and we have not really the simultaneous orthogonality relations (1) with respect to positive measures $\mu_1$ and $\mu_2$. Another approach was proposed in [17]. In this paper we extend the class of [17], obtain the differential equation of third order, study the recurrence relations and precise the ratio asymptotic and zeros distribution of polynomials.

## 2. Polynomials of simultaneous orthogonality

In this section we recall some properties of polynomials of simultaneous orthogonality (see the complete proofs in [21]). First note that for a given vector-index $\bar{n}$ a polynomial $Q_{\bar{n}}$ always exist but is not unique even if we add a normalization condition. There are two general cases where the polynomial is unique (up to constant factor). This is the so-called Angelesko case and Nikishin case. For Angelesko case the supports $\Delta_j$ of measures $\mu_1, \mu_2, \ldots, \mu_p$ are disjoint and one can prove that polynomial $Q_{\bar{n}}$ has exactly $n_j$ simple zeros on the interval $\Delta_j$, $j = 1, 2, \ldots, p$. This implies the following:

**Lemma 1.** *If $\Delta_i \cap \Delta_j = \emptyset$ for $i, j = 1, 2, \ldots, p$ then for all $\bar{n} = (n_1, n_2, \ldots, n_p)$ one has*

$$\int_{\Delta_j} Q_{\bar{n}}(x) x^{n_j} \mathrm{d}\mu_j \neq 0, \quad j = 1, 2, \ldots, p.$$

The statement of lemma means some normality of the table of polynomials and implies the uniqueness property. In this paper we are interested in the case $p = 2$ and in the sequence of polynomials $Q_{\bar{n}}$ associated with the sequence of indexes

$$(0,0), (1,0), (1,1), (2,1), (2,2), (3,2), \ldots \tag{2}$$

In this case the polynomials are defined by only one parameter $n = n_1 + n_2$. We shall denote them by $Q_n$, $n = 0, 1, 2, \ldots$. If $n = 2k$ then $\bar{n} = (k, k)$ and $Q_n = Q_{k,k}$, for $n = 2k + 1$ we have $\bar{n} = (k + 1, k)$ and $Q_n = Q_{k+1,k}$. In this case the Lemma 1 implies the next one:

**Lemma 2.** *The monic polynomials $Q_n$ associated with the sequence of indexes* (2) *satisfy a third order recurrence relation*

$$Q_n(z) = (z + d_n) Q_{n-1}(z) + e_n Q_{n-2}(z) + r_n Q_{n-3}(z). \tag{3}$$

Indeed, suppose $n = 2k$, then $\bar{n} = (k, k)$. If we put $P = (z + d)Q_{n-1} + eQ_{n-2} + rQ_{n-3}$, then $P$ is a polynomial of degree $n$. It is "quasi-orthogonal":

$$\int_{\Delta_1} P(x)x^{\nu}\,\mathrm{d}\mu_1 = 0, \quad \nu = 0, \dots, k-2, \qquad \int_{\Delta_2} P(x)x^{\nu}\,\mathrm{d}\mu_2 = 0, \quad \nu = 0, \dots, k-3.$$

So we have to find the coefficients $d, e, r$ to make $P$ orthogonal to $x^{k-1}$ on $\Delta_1$ and to $x^{k-2}, x^{k-1}$ on $\Delta_2$. It is possible by Lemma 1. By uniqueness we have $P = Q_n$. The proof is the same for $n = 2k + 1$. In general case the recurrence relation is of order $(p + 1)$ [21].

It is not difficult to calculate the coefficients of recurrences (3) from orthogonality relations (1). Let us introduce the notations

$$A_k^{(j)} = \int_{\Delta_j} Q_{k,k}(x)x^k\,\mathrm{d}\mu_j, \quad j = 1, 2, \qquad A_k^{(3)} = \int_{\Delta_1} Q_{k,k}(x)x^{k+1}\,\mathrm{d}\mu_1$$

and

$$B_k^{(j)} = \int_{\Delta_j} Q_{k,k-1}(x)x^{k-j+1}\,\mathrm{d}\mu_j, \quad j = 1, 2; \qquad B_k^{(3)} = \int_{\Delta_2} Q_{k,k-1}(x)x^k\,\mathrm{d}\mu_2.$$

Suppose $n = 2k$, then we have from Lemma 2:

$$Q_{k,k}(z) = (z + d_{2k})Q_{k,k-1}(z) + e_{2k}Q_{k-1,k-1}(z) + r_{2k}Q_{k-1,k-2}(z).$$

To calculate $r_{2k}$ we multiply this relation by $x^{k-2}$ and take the integral over $\Delta_2$. To get $e_{2k}$ we multiply the same relation by $x^{k-1}$ and take the integral over $\Delta_1$. Thus we obtain

$$r_{2k} = -\frac{B_k^{(2)}}{B_{k-1}^{(2)}}, \quad e_{2k} = \left(-\frac{B_k^{(1)}}{B_{k-1}^{(1)}} - r_{2k}\right)\frac{B_{k-1}^{(1)}}{A_{k-1}^{(1)}}. \tag{4}$$

In the same way we get:

$$r_{2k+1} = -\frac{A_k^{(1)}}{A_{k-1}^{(1)}}, \quad e_{2k+1} = \left(-\frac{A_k^{(2)}}{A_{k-1}^{(2)}} - r_{2k+1}\right)\frac{A_{k-1}^{(2)}}{B_k^{(2)}}, \tag{5}$$

$$d_{2k} = \left(\frac{B_{k-1}^{(3)}}{B_{k-1}^{(2)}} - \frac{B_k^{(3)}}{B_k^{(2)}}\right) - \frac{A_{k-1}^{(2)}}{B_k^{(2)}}e_{2k}, \quad d_{2k+1} = \left(\frac{A_{k-1}^{(3)}}{A_{k-1}^{(1)}} - \frac{A_k^{(3)}}{A_k^{(1)}}\right) - \frac{B_k^{(1)}}{A_k^{(1)}}e_{2k+1}. \tag{6}$$

## 3. A systems of "classical" polynomials of simultaneous orthogonality, $p = 2$

We consider two intervals $\Delta_1 = [a, 0]$, $\Delta_2 = [0, 1]$ $(-1 \leqslant a < 0)$ and two measures $\mu_1, \mu_2$ defined by

$$\mathrm{d}\mu_1 = |h(x)|\mathrm{d}x \text{ on } \Delta_1 \quad \text{and} \quad \mathrm{d}\mu_2 = |h(x)|\mathrm{d}x \quad \text{on } \Delta_2$$

where $h(x) = (x - a)^{\alpha}(x - 1)^{\beta}x^{\gamma} (\alpha, \beta, \gamma > -1)$. The intervals $\Delta_1$ and $\Delta_1$ have one common end point $z = 0$. As was shown in [5] this situation is a model for a general one of two disjoint intervals. Let $Q_n = Q_n(\alpha, \beta, \gamma | z)$ be the sequence of polynomials of simultaneous orthogonality associated with the sequence of index (2). Our main tool to study polynomials $Q_n$ is the Rodrigues formula for them. Let $B(z) = z(z-a)(z-1)$ and $n = 2k$, then one can verify that the function

$$F(z) = \frac{1}{h(z)} \frac{\mathrm{d}^k}{\mathrm{d}z^k} [h(z) B^k(z)]$$

is a polynomial of degree $n = 2k$ simultaneously orthogonal with index $(k, k)$ [17, 21]. So for monic polynomial of simultaneous orthogonality $Q_{k,k}$ we have

$$Q_{k,k}(z) = Q_{k,k}(\alpha, \beta, \gamma \mid z) = \frac{1}{M_k} \frac{1}{h(z)} \frac{\mathrm{d}^k}{\mathrm{d}z^k} [h(z) B^k(z)], \tag{7}$$

where $M_k$ is the leading coefficient of polynomial $F(z)$. To calculate $M_k$ we note that $B(z)h'(z) = A(z)h(z)$, where $A(z) := \alpha(z-1)z + \beta(z-a)z + \gamma(z-a)(z-1)$, $\deg A = 2$. Then

$$\frac{\mathrm{d}^j}{\mathrm{d}x^j} [B^k h] = B^{k-j} P_j h$$

with $P_j(x)$ polynomial in $x$. The next derivative in this relation gives

$$P_{j+1} = (k-j) B' P_j + B P_j' + PA$$

It implies $\deg P_j = 2j$ and if $a_j$ is the leading coefficient of $P_j(x)$ then $a_{j+1} = (3k - j + s) a_j$, $P_0 = 1$, $a_0 = 1$. Thus we obtain:

$$M_k = (3k+s)(3k-1+s) \cdots (2k+1+s), \tag{8}$$

where $s = \alpha + \beta + \gamma$. For the case $n = 2k-1$ we have to modify the formula (7). Indeed, one can verify that the function

$$G(z) = \frac{1}{h(z)} \frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}} [h(z) B^{k-1}(z)(z-t)]$$

is a polynomial of degree $n = 2k-1$, quasi-orthogonal with respect to $\mu_1$ and $\mu_2$, that is, $G(x)$ is orthogonal to $1, x, x^2, \ldots, x^{k-2}$ on $\Delta_1$ and on $\Delta_2$. By choosing the constant $t$ properly one can make it really orthogonal. To do it we introduce the notations:

$$C_k^{(j)} = \int_a^0 x^j B^k(x) |h(x)| \, \mathrm{d}x, \quad D_k^{(j)} = \int_0^1 x^j B^k(x) |h(x)| \, \mathrm{d}x, \tag{9}$$

$j = 0, 1, 2$. Then the good choice of the constant $t$ is

$$t = t_k := C_{k-1}^{(1)} / C_{k-1}^{(0)}.$$

Thus for the monic polynomial $Q_{k,k-1}$ we have

$$Q_{k,k-1}(z) = Q_{k,k-1}(\alpha, \beta, \gamma \mid z) = \frac{1}{N_k} \frac{1}{h(z)} \frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}} [h(z) B^{k-1}(z)(z-t_k)], \tag{10}$$

where $N_k$ is the leading coefficient of $G(z)$ and $t_k$ is defined above. One can calculate $N_k$ in the same way as $M_k$ and one has

$$N_k = (3k - 2 + s)(3k - 3 + s) \cdots (2k + s), \tag{11}$$

where as above $s = \alpha + \beta + \gamma$. Note that the polynomials defined by a Rodrigues formula are known in the investigations of polynomials systems [3, 1, 13, 16]. Our contribution is to investigate their properties in connection with simultaneous orthogonality.

**Remarks.** The function $h(z)$ is analytic with the branch points $a$, 0, 1, $\infty$. In formulas (7) and (10) we choose the branches properly. This weight-function also satisfy the generalized Pearson differential equation $B(z)h'(z) = A(z)h(z)$.

## 4. Differential equation

The following theorem may be considered as a justification of term "classical" for polynomials of simultaneous orthogonality $Q_n(\alpha, \beta, \gamma)$:

**Theorem 1.**[2] *For $\alpha = \beta = \gamma = 0$ the polynomial $Q_{k,k}(x)$ satisfies the differential equation*

$$\begin{aligned} &x(x-1)(x-a)Y''' + (6x^2 - 4(a+1)x + 2a)Y'' \\ &\quad + (k-1)(k+2)(-3x + a + 1)Y' - 2k(k^2 + 3k + 2)Y = 0. \end{aligned}$$

**Proof.** The differential equation can be written as

$$BY''' + 2B'Y'' + B''Y' - \frac{k(k+1)}{2}B''Y' - 2k(k+1)(k+2)Y = 0,$$

where as above $B(x) = x(x-a)(x-1)$. Let $y_k = [B^k(x)]^{(k)} = M_k Q_{k,k}$ and

$$P = By_k''' + 2B'y_k'' + B''y_k' - \frac{k(k+1)}{2}B''y_k' = (By_k')'' - \frac{k(k+1)}{2}B''y_k'.$$

Then $P$ is a polynomial of degree $2k$ with leading coefficient $2M_k k(k+1)(k+2)$. If we prove that $P(x)$ is orthogonal to $1, x, x^2, \dots, x^{k-1}$ on $\Delta_1$ and $\Delta_2$ then by uniqueness we can conclude that $P = 2k(k+1)(k+2)y_k$ and theorem follows. One has (integration by part):

$$\int_\Delta (By_k')''x^\nu \, dx = (B'y_k'x^\nu)\Big|_\Delta \quad \text{and} \quad \int_\Delta B''y_k'x^\nu \, dx = (B''y_kx^\nu)\Big|_\Delta$$

[2] We realized, after submitting this paper for publication, that the differential equation in the general case was given in [13].

for $v = 0,1,2,\ldots,k-1$ and $\Delta = \Delta_1, \Delta_2$. It implies for the same $v$:

$$\int_\Delta P(x)x^v\,dx = \left[B'y_k' - \frac{k(k+1)}{2}B''y_k\right]x^v\bigg|_\Delta$$

The orthogonality follows from the following:

**Lemma 3.** *For all* $k \geqslant 1$

$$\left[B'y_k' - \frac{k(k+1)}{2}B''y_k\right]\bigg|_{x=a,0,1} = 0.$$

**Proof.** for $k = 1$ we have $B'y_1' - B''y_1 = B'B'' - B''B' = 0$ so the lemma holds. Suppose by induction that the lemma is valid for $k-1$, that is

$$B'[B^{k-1}]^{(k)} - \frac{k(k-1)}{2}B''[B^{k-1}]^{(k-1)} = 0 \quad \text{for } x = a,0,1.$$

Then we get from the Leibnitz formula:

$$\begin{aligned} y_k &= [BB^{k-1}]^{(k)} = B[B^{k-1}]^{(k)} + C_k^1 B'[B^{k-1}]^{(k-1)} + C_k^2 B''[B^{k-1}]^{(k-2)} \\ &\quad + C_k^3 B'''[B^{k-1}]^{(k-3)}, \\ y_k' &= [BB^{k-1}]^{(k+1)} = B[B^{k-1}]^{(k+1)} + C_{k+1}^1 B'[B^{k-1}]^{(k)} + C_{k+1}^2 B''[B^{k-1}]^{(k-1)} \\ &\quad + C_{k+1}^3 B'''[B^{k-1}]^{(k-2)}. \end{aligned}$$

Polynomials $B(x)$ and $[B^{k-1}(x)]^{(k-j)}$ have zeros at $x = a$, 0, 1 for $j \geqslant 2$ and so

$$\left[B'y_k' - \frac{k(k+1)}{2}B''y_k\right]\bigg|_{x=a,0,1} = \left[B'y_{k-1}' - \frac{k(k-1)}{2}B''y_{k-1}\right]\bigg|_{x=a,0,1} = 0.$$

The lemma and the theorem are proved. □

## 5. Recurrence relations

In this section we study the coefficients of recurrence relations (3) for the introduced polynomials $Q_n(z) = Q_n(\alpha,\beta,\gamma\,|\,z)$. Our conclusions is:

**Theorem 2.** *The recurrence relations for* $Q_n(\alpha,\beta,\gamma\,|\,z)$ *are limit periodic with period* 2, *more precisely*

$$d_{2k} \to d^{(2)} := -\frac{a+1}{9} - \frac{2}{3}x_2, \quad d_{2k+1} \to d^{(1)} := -\frac{a+1}{9} - \frac{2}{3}x_1,$$

$$e_{2k} \to e^{(2)} := -\frac{4}{81}(a^2 - a + 1), \quad e_{2k+1} \to e^{(1)} := -\frac{4}{81}(a^2 - a + 1),$$

$$r_{2k} \to r^{(2)} := \frac{4}{27}B(x_2), \quad r_{2k+1} \to r^{(1)} := \frac{4}{27}B(x_1),$$

*where* $x_1, x_2$ *are the solutions of the equation* $B'(x) = 0$ *with* $a < x_1 < 0$ *and* $0 < x_2 < 1$.

**Proof.** Using the notations (9) and formulas (7) and (10) we can calculate $A_k^{(j)}$, $B_k^{(j)}$, $j = 1, 2, 3$:

$$A_k^{(1)} = \frac{(-1)^k k!}{M_k} C_k^{(0)}, \quad A_k^{(2)} = \frac{(-1)^k k!}{M_k} D_k^{(0)}, \quad A_k^{(3)} = \frac{(-1)^k (k+1)!}{2M_k} C_k^{(2)},$$

$$B_k^{(1)} = \frac{(-1)^{k-1} k!}{N_k} \frac{C_{k-1}^{(0)} C_{k-1}^{(2)} - C_{k-1}^{(1)} C_{k-1}^{(1)}}{C_{k-1}^{(0)}},$$

$$B_k^{(2)} = \frac{(-1)^{k-1} (k-1)!}{N_k} \frac{D_{k-1}^{(1)} C_{k-1}^{(0)} - D_{k-1}^{(0)} C_{k-1}^{(1)}}{C_{k-1}^{(0)}},$$

$$B_k^{(3)} = \frac{(-1)^{k-1} k!}{N_k} \frac{C_{k-1}^{(0)} D_{k-1}^{(2)} - C_{k-1}^{(1)} D_{k-1}^{(1)}}{C_{k-1}^{(0)}}.$$

This allow us to get the asymptotic of recurrence coefficients. We need the following lemma:

**Lemma 4.** *The following asymptotics holds* ($j = 0, 1, 2$)

$$C_k^{(j)} \asymp x_1^j \, |h(x_1)| \, [B(x_1)]^k \left( \frac{2\pi B(x_1)}{-kB''(x_1)} \right)^{1/2},$$

$$D_k^{(j)} \asymp x_2^j \, |h(x_2)| \, [B(x_2)]^k \left( \frac{2\pi B(x_2)}{-kB''(x_2)} \right)^{1/2},$$

*where as above* $x_1, x_2$ *are the solutions of the equation* $B'(x) = 0$ *with* $a < x_1 < 0$ *and* $0 < x_2 < 1$.

**Proof.** We use the Laplace method for asymptotics of integrals (see [9]). We have for example:

$$C_k^{(j)} := \int_a^0 x^j B^k(x) \, |h(x)| \, dx = B^k(x_1) \int_a^0 x^j \, |h(x)| \exp[k \log(B(x)/B(x_1))] \, dx.$$

The functions $g_j(x) = x^j |h(x)|$, $f(x) = \log(B(x)/B(x_1))$ satisfy the conditions of Laplace method ([9, p. 66]) and we get

$$C_k^{(j)} = B^k(x_1) \left[ \frac{d_0^{(j)}}{\sqrt{k}} + \frac{d_1^{(j)}}{k\sqrt{k}} + o\left( \frac{1}{k^{3/2}} \right) \right]. \tag{12}$$

For the coefficients of asymptotic $d_0^{(j)}$, $d_1^{(j)}$ we use the formulas [9]:

$$d_0^{(j)} = \sqrt{\frac{\pi}{-a_2}} \, b_0^{(j)},$$

$$d_1^{(j)} = \frac{1}{(-a_2)^{5/2}} [b_2^{(j)} a_2^2 \Gamma(3/2) - a_2(a_3 b_1^{(j)} + a_4 b_0^{(j)}) \Gamma(5/2) + (1/2) b_0^{(j)} a_3^2 \Gamma(7/2)],$$

where $f(x) = a_2(x - x_1)^2 + a_3(x - x_1)^3 + a_4(x - x_1)^4 + \cdots$ and $g_j(x) = b_0^{(j)} + b_1^{(j)}(x - x_1) + b_2^{(j)}(x - x_1)^2 + \cdots$ The same is valid for $D_k^{(j)}$, we have to replace $[a, 0]$ by $[0, 1]$ and $x_1$ by $x_2$. The lemma follows after a correct calculation. □

To get the limits of recurrence coefficients we apply the formulas (4)–(6), (8), (11) and the asymptotics of $C_k^{(j)}$, $D_k^{(j)}$. Really the first term of asymptotic in (12) is sufficient to calculate the limits, only for $e_{2k}$ we use the second term and the following relation

$$C_k^{(0)}C_k^{(2)} - C_k^{(1)}C_k^{(1)} = \frac{4\pi B^{2k+2}(x_1)}{[B''(x_1)]^2}|h(x_1)|^2\left(\frac{1}{2k^2} + \mathrm{o}\left(\frac{1}{k^2}\right)\right).$$

The theorem is proved. Note that the limits of recurrence coefficients do not depend on $\alpha, \beta, \gamma$. □

**Remark.** For $a = -1$ the statement of theorem was used in [6].

## 6. Asymptotic of ratio. Zeros distribution

For general Angelesko system the distribution of zeros of polynomials of simultaneous orthogonality was investigated in [14]. The measures of distribution of zeros are the solutions of some equilibrium problem for their complex logarithmic potentials (see [14, 21]). As was shown in [5] this equilibrium potential problem can be reduced to investigation of an associated algebraic function. The strong or power asymptotics of polynomials of simultaneous orthogonality was studied in [4, 7, 23]. The method of investigation is based on some extremal problems in the Hilbert spaces of analytic functions. All these results are valid for the system $\{Q_n(\alpha, \beta, \gamma \mid z)\}$. In this section we use Theorem 2 to precise the asymptotics of ratio $Q_{n+1}/Q_n$. First note that we can obtain from (3) the third order recurrence for polynomials $Q_n, Q_{n-2}, Q_{n-4}, Q_{n-6}$:

$$Q_n(z) = R_n^{(1)}(z)Q_{n-2} + R_n^{(2)}(z)Q_{n-4} + R_n^{(3)}(z)Q_{n-6}, \tag{13}$$

where $R_n^{(j)}, j = 1, 2, 3$ are rational functions in $z$. The advantage of these recurrences is that there are limit periodic with period 1. More precisely, using Theorem 2 one can verify that $\forall z \in C$

$$\lim R_n^{(1)} = P_1(z) := [z^2 - (2/3)(1 + a)z - (1/27)(a^2 - 10a + 1)],$$

$$\lim R_n^{(2)} = P_2(z) := (2/9)^3[(a^3 - 4a^2 + a) + (-2 + 3a + 3a^2 - 2a^3)z],$$

$$\lim R_n^{(3)} = P_3(z) := -2(2/27)^3(a^2 - 2a^3 + a^4).$$

Thus, one can associate with (13) the following algebraic function

$$w^3 - P_1(z)w^2 - P_2(z)w - P_3 = 0. \tag{14}$$

The function $w(z)$ has the branch points at $a$, 0, 1, and at the point $z_a$, where [18]

$$z_a = \frac{1}{9}\frac{(a + 1)^3}{a^2 - a + 1}.$$

All these branch points are of second order. One branch of $w(z)$ has a poles of order 2 at infinity and two others branches have there a simple zeros. If we denote its by $w_0(z)$, $w_1(z)$, and $w_3(z)$ then $w_0(z)$ is meromorphic in extended complex plane $\bar{C}$ with the cuts over $[a, 0]$ and $[z_a, 1]$, $w_1(z)$ is holomorphic in $\bar{C}$ with the cut over $[a, 0]$ and $w_2(z)$ is holomorphic in $\bar{C}$ with the cut over $[z_a, 1]$. All these facts can be proved by analysis of equation for $w$ (see [17]). Now we are able to state.

**Theorem 3.** $\forall z \notin [a,0] \cup [z_a, 1]$,

$$\lim_{k\to\infty} \frac{Q_{k,k}(z)}{Q_{k-1,k-1}(z)} = \lim_{k\to\infty} \frac{Q_{k+1,k}(z)}{Q_{k,k-1}(z)} = w_0(z)$$

*and*

$$\lim_{k\to\infty} \frac{Q_{2k}}{Q_{2k-1}} = \frac{(z+d^{(2)})w_0(z)+r^{(2)}}{w_0(z)+e^{(2)}}, \quad \lim_{k\to\infty} \frac{Q_{2k+1}}{Q_{2k}} = \frac{(z+d^{(1)})w_0(z)+r^{(1)}}{w_0(z)+e^{(1)}}.$$

**Proof.** To prove the first relation we can use the result on asymptotics of polynomials of simultaneous orthogonality from [4] (really we use only one part of the general result). According to this result there is an algebraic function of third order $\Phi(z)$, such that outside the interval $[a, 1]$ one has

$$\lim_{k\to\infty} \frac{Q_{k,k}(z)}{Q_{k-1,k-1}(z)} = \Phi_0(z),$$

where $\Phi_1(z)$ is the branch of $\Phi(z)$ with the double pole at infinity. If we go to limit ($n \to \infty$) in the recurrence (13) we obtain immediately

$$\Phi_0^3 - P_1\Phi_0^2 - P_2\Phi_0 - P_3 = 0.$$

So $\Phi_0(z) = w_0(z)$ and the statement follows. In the same way we obtain the asymptotics of ratio $Q_{n+2}/Q_n$, $n$ is odd. For the second line of asymptotics of the theorem we use the recurrence (3) and Theorem 2. □

According to general result of [14] the zeros of polynomials $Q_{k,k}$ and $Q_{k+1,k}$ are concentrated on two intervals $[a,0]$ and $[z_a, 1]$. The zeros distribution does not depend on $\alpha, \beta, \gamma$ and it is defined by the functions $w_1$ and $w_2$ on intervals $[a,0]$ and $[0,1]$ respectively. Note that the limit of ratio $Q_{n+1}/Q_n$ does not exist. Eq. (14) was already introduced in [18] and used in [5] in connection with some vector continued fractions.

**Remark 1.** It is possible to get the exact asymptotics of polynomials $Q_{k,k}$, $Q_{k,k-1}$ using the Rodrigues formulas (7), (10) and Darboux method (see [17] for the case $a = -1$). The result is of the same form as in the general case:

$$Q_{k,k}(z) = C_k[w_0(z)]^k F(z)[1+\varepsilon_1(z)], \quad Q_{k+1,k}(z) = B_k[w_0(z)]^k G(z)[1+\varepsilon_2(z)],$$

where $C_k$, $B_k$ are a constants, $F(z), G(z)$ are analytic in the exterior of two intervals $[a,0]$ and $[z_a, 1]$. These functions can be calculated exactly and thus the full asymptotics for our classical systems of polynomials may be given. In this section we restricted our attention to the asymptotics of ratio of polynomials by simplicity of presentation.

**Remark 2.** It is interesting to compare the results, on asymptotics of ratio with Poincaré theorem for recurrence equations (see [15]). The statement of Poincaré theorem is essentially the following: If the coefficients $a_n^{(j)}$ of recurrence equation

$$Y_n + a_n^{(1)}Y_{n-1} + a_n^{(2)}Y_{n-2} + \cdots + a_n^{(d)}Y_{n-d} = 0$$

have a limit $a_n^{(j)} \to a^{(j)}$, $j = 1, 2, \ldots, d$ and all zeros of characteristic equation

$$w^d + a^{(1)}w^{d-1} + a^{(2)}w^{d-2} + \cdots + a^{(d)} = 0 \tag{15}$$

are of different modulus, then for each solution $Y_n$ there is a limit of ratio

$$\lim_{n\to\infty} \frac{Y_n}{Y_{n-1}} = w,$$

where $w$ is one of the zeros of algebraic equation (15). There are two difficulties of application of this theorem in our case. First is the choice of zero $w$ of associated algebraic equation (that is the choice of the branch of associated algebraic function). The second is connected with the curves of equal modulus of the algebraic function $w(z)$. Our conjecture is that in the case of limit periodic recurrences of the type (3) the equality $\lim Y_{n+1}(z)/Y_n(z) = w_j(z)$, $n \in \Lambda$ with the same branch $w_j(z)$ holds on the resolvent set of associated difference operator.

## Acknowledgements

The authors express the sincere gratitude to the referee for his useful criticism.

## References

[1] N. Abramesko, Sulle sere di poling hi una variabile compress. Le sere de Darboux, *Ann. Math. Pure Appl.* **31** (1922) 207–249.
[2] W.A. Al-Salam, Characterization theorems for orthogonal polynomials, in: P. Nevai, Ed., *Orthogonal Polynomials: Theory and Practice* (Kluwer, Dordrecht, 1990) 1–24.
[3] P. Appell, Sur une suite de polynômes ayant toutes leurs racines réelles, *Arch. Math. Phys.* **1** (1901) 69–71.
[4] A.I. Aptekarev, Asymptotics of polynomials with $p$ relations of orthogonality, *Math. Sbornik* **136** (1988) 56–84.
[5] A.I. Aptekarev and V.A. Kaliaguine, $N$-th root asymptotics of polynomials of simultaneous orthogonality, *Inst. Keldych of Appl. Math., USSR Acad. Scien.*, preprint IAM 60 (1986).
[6] A.I. Aptekarev and V.A. Kaliaguine, Analytic properties of two-dimentional continued $P$-fraction expansions with periodical coefficients and their simultaneous Padé–Hermite approximants, in: J. Gilewicz et al., Eds., *Rational Approximations and its Applications*, Proceedings, Lancut 1985, Lecture Notes in Math., Vol. 1237 (Springer, Berlin, 1987) 145–160.
[7] A.I. Aptekarev and H. Stahl, Asymptotics of Hermite–Padé polynomials, in: *Progress in Approximation Theory*, Springer Series in Comp. Math., Vol. 19 (Springer, Berlin, 1992) 127–169.
[8] T.S. Chirara, The three term recurrence relation and spectral properties of orthogonal polynomials, in: P. Nevai, Ed., *Orthogonal Polynomials: Theory and Practice* (Kluwer Academic Publisher, Dordrecht, 1990) 99–114.
[9] N.G. De Bruijn, *Asymptotic Methods in Analysis* (North-Holland, Amsterdam, 1961).
[10] M.G. de Bruin, Simultaneous Padé approximants and orthogonality, Lecture Notes in Math., Vol. 1171 (Springer, Berlin, 1985) 74–83.
[11] K. Douak and P. Maroni, Les polinômes orthogonaux "Classiques" de dimension deux, Université Pierre et Marie Curie, Lab. d'analyse Numérique, 1990.
[12] A. Draux and A. Maanaoui, Vector orthogonal polynomials, *J. Comput. Appl. Math.* **32** (1990) 59–68.
[13] G.K. Engelis, On polynomials defined by the Rodrigues formula, *Uchen. Zap. Latv. Gos. Univ.* **20** (1958) 137–143 (in Russian).
[14] A.A. Gontchar and E.A. Rakhmanov, On the convergence of simultaneoues Pade approximants for a systems of Markov type functions, *Proc. Steklov Math. Inst.*, Issue 3 (1983).

[15] A.O. Guelfond, *Calculus of Finite Differences* (Physmathgis, Moscow, 1961) (in Russian). French translation: *Calcul des Différences Finies* (Dunod, Paris, 1963).

[16] P. Humbert, Sur les équations de Didon, *Nouvelles Annales de Math.* **19** (1919) 443–451.

[17] V.A. Kaliaguine, On a classe of polynomials defined by two orthogonality relations, *Math. Sbornik* **110** (1979) 609–627. English transl. in *Math. URSS Sb.* **38** (1981) 563–580.

[18] V.A. Kaliaguine, On simultaneous Padé approximants of two logarithmic functions, in: S. Nikolski, Ed., *Theory of Functions and Approximation* (Saratov, 1986) Vol. 2, 126–130.

[19] V.A. Kaliaguine, Les polynômes orthogonaux de dimention d et les approximants de Padé–Hermite, Seminaire AMS de Rouen, Rouen, 1993.

[20] T.H. Koornwinder, Orthogonal polynomials in connection with quantum groups, in: P. Nevai, Ed., *Orthogonal Polynomials: Theory and Practice* (Kluwer, Dordrecht, 1990) 257–292.

[21] E.M. Nikishin and V.N. Sorokin, *Rational approximations and orthogonality* (Nauka, Moscow, 1988); Transl. Math. Monographs, Vol. 92 (Amer. Math. Soc., Providence, RI, 1991).

[22] P.K. Souetin, *Classical Orthogonal Polynomials* (Nauka, Moscow, 1986) (in Russian).

[23] H. Stahl, Asymptotics of Hermite–Padé polynomials and related approximants, summary of results, Reprint, Berlin, 1987.

[24] J. Van Iseghem, Approximants de Padé vectoriels, Thèse, Université de Sciences et Techniques de Lille, 1987.